

Non-linear relativistic perturbation theory with two parameters

Carlos F. Sopuerta¹, Marco Bruni¹ and Leonardo Gualtieri²

¹*University of Portsmouth, United Kingdom.*

²*Università di Roma "La Sapienza" and Sezione INFN ROMA, Italy.*

An underlying fundamental assumption in relativistic perturbation theory is the existence of a parametric family of spacetimes that can be Taylor expanded around a background. Since the choice of the latter is crucial, sometimes it is convenient to have a perturbative formalism based on two (or more) parameters. A good example is the study of rotating stars, where generic perturbations are constructed on top of an axisymmetric configuration built by using the slow rotation approximation. Here, we discuss the gauge dependence of non-linear perturbations depending on two parameters and how to derive explicit higher order gauge transformation rules.

1 Introduction

Relativistic perturbation theory is based on the assumption that there exists a parametric family of spacetimes that can be Taylor expanded around a certain background. The perturbations are then defined as the derivative terms of this series, evaluated on that background. In most cases of interest one deals with an expansion in a single parameter, which can either be a formal one, as in cosmology [1], or can have a specific physical meaning, as in the study of binary black hole mergers via the close limit approximation [2]. In some physical applications it may be instead convenient to construct a perturbative formalism based on two (or more) parameters, because the choice of background is crucial in having a manageable theory. The study of perturbations of stationary axisymmetric rotating stars is a good example. In this case, an analytic stationary axisymmetric solution is not known, at least for reasonably interesting equations of state. A common procedure is to treat axisymmetric stars using the so-called slow rotation approximation, so that the background is a star with spherical symmetry. In this approach the first order in Ω discloses frame dragging effects, with the star actually remaining spherical; Ω^2 terms carry the effects of rotation on the fluid. Generic time dependent perturbations of the rotating star (parametrized by a dummy parameter λ and describing oscillations) are then built on top of the stationary axisymmetric perturbations in Ω . Clearly, in this approach any interesting physics requires non-linear perturbations, as at least terms of order $\lambda\Omega$ need to be considered. Here, we discuss the formulation of non-linear relativistic perturbation theory with two parameters. Details can be found in [3].

2 Two-parameter perturbation theory

We start with the introduction of an adequate framework for the formulation of perturbation theory with two parameters. Following the spirit of [4] we introduce an $(m+2)$ -dimensional manifold \mathcal{N} , foliated by m -dimensional submanifolds diffeomorphic to \mathcal{M} , the spacetime manifold, so that $\mathcal{N} = \mathcal{M} \times \mathbb{R}^2$. We shall label each copy of \mathcal{M} by the corresponding value of the parameters λ, Ω . The manifold \mathcal{N} has a natural differentiable structure which is the direct product of those of \mathcal{M} and \mathbb{R}^2 . We can then choose charts on \mathcal{N} in which x^μ ($\mu = 0, 1, \dots, m-1$) are coordinates of each leaf $\mathcal{M}_{\lambda, \Omega}$ and $x^m = \lambda$, $x^{m+1} = \Omega$.

If a tensor field $T_{\lambda, \Omega}$ is given on each $\mathcal{M}_{\lambda, \Omega}$, we have that a tensor field T is automatically defined on \mathcal{N} by the relation $T(p, \lambda, \Omega) := T_{\lambda, \Omega}(p)$, with $p \in \mathcal{M}_{\lambda, \Omega}$. In particular, on each $\mathcal{M}_{\lambda, \Omega}$ one has a metric $g_{\lambda, \Omega}$ and a set of matter fields $\tau_{\lambda, \Omega}$, satisfying a set of field equations

$$\mathcal{E}[g_{\lambda, \Omega}, \tau_{\lambda, \Omega}] = 0,$$

which we do not need to specify since this formulation of perturbation theory can be applied to any spacetime theory.

We now want to define the perturbation in any tensor field T , therefore we must find a way to compare $T_{\lambda, \Omega}$ with the background value T_0 . This requires a prescription for identifying points of $\mathcal{M}_{\lambda, \Omega}$ with those of \mathcal{M}_0 . This is easily accomplished by assigning a diffeomorphism $\varphi_{\lambda, \Omega} : \mathcal{N} \rightarrow \mathcal{N}$ such that $\varphi_{\lambda, \Omega}|_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_{\lambda, \Omega}$. Clearly, $\varphi_{\lambda, \Omega}$ can be regarded as the member of a two-parameter group of diffeomorphisms φ on \mathcal{N} , corresponding to the values of λ, Ω of the group parameter. Therefore, we could equally well give the vector fields ${}^\varphi\eta$, ${}^\varphi\zeta$ that generate φ . In our construction the group of diffeomorphisms is chosen to be Abelian and therefore ${}^\varphi\eta$ and ${}^\varphi\zeta$ commute. In this way, perturbing first with respect to the parameter λ and afterwards with respect to Ω coincides with the converse operation.

The choice of the group of diffeomorphisms φ , or equivalently, the choice of the vector fields ${}^\varphi\eta$, ${}^\varphi\zeta$, is what in perturbation theory is called a *gauge choice*. Using the pull-back of φ , the perturbation in T can now be defined simply as

$$\Delta_0 T_{\lambda, \Omega} := \varphi_{\lambda, \Omega}^* T|_{\mathcal{M}_0} - T_0. \quad (1)$$

The first term on the right-hand side can be Taylor-expanded to get [3]

$$\Delta_0 T_{\lambda, \Omega} = \sum_{k, k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \delta_{\varphi}^{(k, k')} T - T_0, \quad (2)$$

where

$$\delta_{\varphi}^{(k, k')} T := \left[\frac{\partial^{k+k'}}{\partial \lambda^k \partial \Omega^{k'}} \varphi_{\lambda, \Omega}^* T \right]_{\lambda=0, \Omega=0, \mathcal{M}_0} = \mathcal{L}_{{}^\varphi\eta}^k \mathcal{L}_{{}^\varphi\zeta}^{k'} T|_{\mathcal{M}_0}, \quad (3)$$

and here \mathcal{L} denotes Lie differentiation. This last expression defines the perturbation of order (k, k') of T (notice that $\delta_\varphi^{(0,0)}T = T_0$). It is worth noticing $\Delta_0 T_{\lambda,\Omega}$ and $\delta_\varphi^{(k,k')}T$ are defined on \mathcal{M}_0 ; this formalizes the statement one commonly finds in the literature that *perturbations are fields living in the background*. It is important to appreciate that the parameters λ, Ω labelling the various spacetime models also serve to perform the expansion (2), and therefore determine what one means by *perturbations of order (k, k')* .

3 Gauge invariance and gauge transformations

Let us now suppose that two gauges φ and ψ , described by pairs of vector fields $({}^\varphi\eta, {}^\varphi\zeta)$ and $({}^\psi\eta, {}^\psi\zeta)$ respectively, are defined on \mathcal{N} . Correspondingly, the integral curves of $({}^\varphi\eta, {}^\varphi\zeta)$ and $({}^\psi\eta, {}^\psi\zeta)$ define two two-parameter groups of diffeomorphisms φ and ψ on \mathcal{N} , that connect any two leaves of the foliation. Thus, $({}^\varphi\eta, {}^\varphi\zeta)$ and $({}^\psi\eta, {}^\psi\zeta)$ are everywhere transverse to $\mathcal{M}_{\lambda,\Omega}$ and points lying on the same integral surface of either of the two are to be regarded *as the same point* within the respective gauge: φ and ψ are both point identification maps, i.e. two different gauge choices.

The pairs of vector fields $({}^\varphi\eta, {}^\varphi\zeta)$ and $({}^\psi\eta, {}^\psi\zeta)$ can both be used to pull back a generic tensor T and therefore to construct two other tensor fields $\varphi_{\lambda,\Omega}^*T$ and $\psi_{\lambda,\Omega}^*T$, for any given value of (λ, Ω) . In particular, on \mathcal{M}_0 we now have three tensor fields, i.e. T_0 and

$$T_{\lambda,\Omega}^\varphi := \varphi_{\lambda,\Omega}^*T|_{\mathcal{M}_0}, \quad T_{\lambda,\Omega}^\psi := \psi_{\lambda,\Omega}^*T|_{\mathcal{M}_0}.$$

Since φ and ψ represent gauge choices for mapping a perturbed manifold $\mathcal{M}_{\lambda,\Omega}$ into the unperturbed one \mathcal{M}_0 , $T_{\lambda,\Omega}^\varphi$ and $T_{\lambda,\Omega}^\psi$ are the representations, in \mathcal{M}_0 , of the perturbed tensor according to the two gauges. Using (1–3) we can write

$$T_{\lambda,\Omega}^\varphi = \sum_{k=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \delta_\varphi^{(k,k')} T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k! k'!} \mathcal{L}_{{}^\varphi\eta}^k \mathcal{L}_{{}^\varphi\zeta}^{k'} T = T_0 + \Delta_0^\varphi T_{\lambda,\Omega}, \quad (4)$$

and the analogous one for $T_{\lambda,\Omega}^\psi$. In this expression $\delta_\varphi^{(k,k')}T$ denotes the perturbations (3) in the gauge φ .

3.1 Gauge invariance

Given a tensor field T , if $T_{\lambda,\Omega}^\varphi = T_{\lambda,\Omega}^\psi$, for any pair of gauges φ and ψ , we say that T is *totally gauge invariant*. In general this is a very strong condition, because then (4) and the analogous equation in the gauge ψ would imply that $\delta_\varphi^{(k,k')}T = \delta_\psi^{(k,k')}T$, for any two gauges φ and ψ and for any order (k, k') . However, in practice one is

interested in perturbations up to a fixed order. It is then convenient to weaken the definition given above, saying that T is *gauge invariant up to order (n, n')* iff for any two gauges φ and ψ we have

$$\delta_\varphi^{(k, k')} T = \delta_\psi^{(k, k')} T \quad \forall (k, k') \text{ with } k \leq n, k' \leq n'.$$

From this definition and the definition of perturbation of order (k, k') we have that a tensor field T is gauge invariant to order (n, n') iff in any gauge φ we have that $\mathcal{L}_\xi \delta_\varphi^{(k, k')} T = 0$, for any vector field ξ defined on \mathcal{M} and for any $(k, k') < (n, n')$. As a consequence, T is gauge invariant to order (n, n') iff T_0 and all its perturbations of order lower than (n, n') are, in any gauge, either vanishing or constant scalars, or a combination of Kronecker deltas with constant coefficients.

3.2 Gauge transformations

When a tensor field T is not gauge invariant, it is important to know how its representation on \mathcal{M}_0 changes under a gauge transformation. To this purpose, given two gauges φ and ψ , it is natural to introduce, for each value of $(\lambda, \Omega) \in \mathbb{R}^2$, the diffeomorphism $\Phi_{\lambda, \Omega} : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ defined by

$$\Phi_{\lambda, \Omega} := \varphi_{\lambda, \Omega}^{-1} \circ \psi_{\lambda, \Omega} = \varphi_{-\lambda, -\Omega} \circ \psi_{\lambda, \Omega},$$

which represents the gauge transformation from the gauge φ to the gauge ψ . Indeed, the tensor fields $T_{\lambda, \Omega}^\varphi$ and $T_{\lambda, \Omega}^\psi$, defined on \mathcal{M}_0 by the gauges φ and ψ , are connected by the linear map $\Phi_{\lambda, \Omega}^*$, the pull-back of $\Phi_{\lambda, \Omega}$:

$$\begin{aligned} T_{\lambda, \Omega}^\psi &= \psi_{\lambda, \Omega}^* T|_{\mathcal{M}_0} = (\psi_{\lambda, \Omega}^* \varphi_{-\lambda, -\Omega}^* \varphi_{\lambda, \Omega}^* T)|_{\mathcal{M}_0} \\ &= \Phi_{\lambda, \Omega}^* (\varphi_{\lambda, \Omega}^* T)|_{\mathcal{M}_0} = \Phi_{\lambda, \Omega}^* T_{\lambda, \Omega}^\varphi. \end{aligned} \quad (5)$$

We must stress that $\Phi : \mathcal{M}_0 \times \mathbb{R}^2 \rightarrow \mathcal{M}_0$ thus defined, *does not* constitute a two-parameter group of diffeomorphisms in \mathcal{M}_0 . In other words, the action of Φ cannot be generated by a pair of vector fields. However, one can show [3] that for any tensor field T , $\Phi_{\lambda, \Omega}^* T$ can be expanded up to total order n (that is, including all the terms of order (k, k') with $k + k' \leq n$) by using $n(n+1)/2$ vector fields. Up to second order, the explicit form of the expansion of equation (5) is (details and the fourth-order expansion are given in [3])

$$\begin{aligned} T_{\lambda, \Omega}^\psi &= T_{\lambda, \Omega}^\varphi + \lambda \mathcal{L}_{\xi(1,0)} T_{\lambda, \Omega}^\varphi + \Omega \mathcal{L}_{\xi(0,1)} T_{\lambda, \Omega}^\varphi \\ &+ \frac{\lambda^2}{2} \left\{ \mathcal{L}_{\xi(2,0)} + \mathcal{L}_{\xi(1,0)}^2 \right\} T_{\lambda, \Omega}^\varphi + \frac{\Omega^2}{2} \left\{ \mathcal{L}_{\xi(0,2)} + \mathcal{L}_{\xi(0,1)}^2 \right\} T_{\lambda, \Omega}^\varphi \\ &+ \lambda \Omega \left\{ \mathcal{L}_{\xi(1,1)} + \epsilon_0 \mathcal{L}_{\xi(1,0)} \mathcal{L}_{\xi(0,1)} + \epsilon_1 \mathcal{L}_{\xi(0,1)} \mathcal{L}_{\xi(1,0)} \right\} T_{\lambda, \Omega}^\varphi + O^3(\lambda, \Omega), \end{aligned}$$

where the $\xi_{(p,q)}$ are the vector fields generating the gauge transformation $\Phi_{\lambda,\Omega}$ and (ϵ_0, ϵ_1) are two real constants such that $\epsilon_0 + \epsilon_1 = 1$. They represent the freedom we have in the reconstruction of the gauge transformation.

With all these ingredients, we can now relate the perturbations in the two gauges φ and ψ . To second total order, these relations are given by

$$\delta_\psi^{(1,0)}T - \delta_\varphi^{(1,0)}T = \mathcal{L}_{\xi_{(1,0)}}T_0, \quad \delta_\psi^{(0,1)}T - \delta_\varphi^{(0,1)}T = \mathcal{L}_{\xi_{(0,1)}}T_0, \quad (6)$$

$$\delta_\psi^{(2,0)}T - \delta_\varphi^{(2,0)}T = 2\mathcal{L}_{\xi_{(1,0)}}\delta_\varphi^{(1,0)}T + \left\{ \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}_{\xi_{(1,0)}}^2 \right\} T_0, \quad (7)$$

$$\begin{aligned} \delta_\psi^{(1,1)}T - \delta_\varphi^{(1,1)}T &= \mathcal{L}_{\xi_{(1,0)}}\delta_\varphi^{(0,1)}T + \mathcal{L}_{\xi_{(0,1)}}\delta_\varphi^{(1,0)}T \\ &+ \left\{ \mathcal{L}_{\xi_{(1,1)}} + \epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + \epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}} \right\} T_0, \end{aligned}$$

$$\delta_\psi^{(0,2)}T - \delta_\varphi^{(0,2)}T = 2\mathcal{L}_{\xi_{(0,1)}}\delta_\varphi^{(0,1)}T + \left\{ \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}_{\xi_{(0,1)}}^2 \right\} T_0,$$

This result is, of course, consistent with the characterization of gauge invariance above, in the subsection 3.1. Equation (6) implies that T is gauge invariant to the order $(1,0)$ or $(0,1)$ iff $\mathcal{L}_\xi T_0 = 0$ for any vector field ξ on \mathcal{M}_0 . Equation (7) implies that T is gauge invariant to the order $(2,0)$ iff $\mathcal{L}_\xi T_0 = 0$ and $\mathcal{L}_\xi \delta_\varphi^{(1,0)}T = 0$ for any vector field ξ on \mathcal{M}_0 , and so on for all the orders.

Finally, we mention that it is also possible to find the explicit expressions for the generators $\xi_{(p,q)}$ of the gauge transformation Φ in terms of the vector fields $({}^\varphi\eta, {}^\varphi\zeta)$ and $({}^\psi\eta, {}^\psi\zeta)$ associated with the gauge choices φ and ψ respectively. Their expressions up to second total order is [3]:

$$\xi_{(1,0)} = {}^\psi\eta - {}^\varphi\eta, \quad \xi_{(0,1)} = {}^\psi\zeta - {}^\varphi\zeta,$$

$$\xi_{(2,0)} = [{}^\varphi\eta, {}^\psi\eta], \quad \xi_{(1,1)} = \epsilon_0[{}^\varphi\eta, {}^\psi\zeta] + \epsilon_1[{}^\varphi\zeta, {}^\psi\eta], \quad \xi_{(0,2)} = [{}^\varphi\zeta, {}^\psi\zeta].$$

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